SHORT COMMUNICATIONS

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Alternative formulae for the number of sublattices

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Abstract

Two formulae for the number of sublattices of a given index k of an n-dimensional lattice are presented. They are based on the decomposition of the index k into a product of prime numbers and have the form of a rational function of these primes. Compared with other known methods, they can give the result in a much quicker and more comfortable way.

1. Introduction

To determine the number $F(n, k)$ of sublattices of the index k† ($k > 0$) of an n-dimensional lattice L ($n > 0$) belongs undoubtedly to the first tasks of lattice theory. Moreover, it can be useful in crystallography, *e.g.* when considering subgroups of the space groups. However, it has not penetrated into the text books on crystallography, not even into *International Tables for Crystallography* (1995) and must be looked for in other regions of mathematics. Thanks to some information that we have been kindly given, we became acquainted with two ways of determining this number.

(i) Cassels (1971) has shown that $F(n, k)$ is equal to the number of matrices of a certain shape, however, he did not suggest how to count them effectively. These matrices are square matrices of rank n with integer elements r_{ij} fulfilling

$$\begin{aligned} 0 &= r_{ij} && \text{for } 1 \le i < j \le n, \\ 0 &\le r_{ij} < r_{jj} && \text{for } 1 \le j < i \le n, \\ r_{11} r_{22} &\ldots r_{nn} = k. && \end{aligned} \tag{1}$$

The set of all such matrices is further denoted $\mathcal{R}(n, k)$.

(ii) Baake (1997), using the result of Scheja & Storch (1988), established a concise formula

$$F(n, k) = \sum d_1^0 d_2^1 \ldots d_n^{n-1} \tag{2}$$

where the sum is over all sequences

$$d_1, d_2, \ldots, d_n$$

of positive integers fulfilling

$$d_1 d_2 \ldots d_n = k.$$

In this paper, we derive two formulae alternative to (2). Although they look more complicated, they are usually quicker for numerical calculations. An example will show this. The formulae do not include the trivial case $k = 1$. Our approach is based on counting the matrices of $\mathcal{R}(n, k)$.

† Meaning that any of these sublattices has the dimension n and is k times 'thinner' than the lattice L.

2. Formulae

Let $n > 0$, $k > 1$ be integers,

$$k = p_1^{q_1} \ldots p_m^{q_m} \quad (m \ge 1), \tag{3}$$

where $p_1, \ldots, p_m$ are mutually different prime numbers and $q_1, \ldots, q_m$ positive integers. Let L be an n-dimensional lattice and $F(n, k)$ the number of sublattices of L of the index k. Then,

$$F(n, k) = \prod_{i=1}^{m} \prod_{j=1}^{q_i} (p_i^{j+n-1} - 1)(p_i^j - 1)^{-1}$$

and also‡

$$F(n, k) = \prod_{i=1}^{m} \prod_{j=1}^{n-1} (p_i^{j+q_i} - 1)(p_i^j - 1)^{-1}. \tag{4}$$

Remark 1. Both formulae may seem rather complicated but they can be reformulated into a simple rule:

For any factor p^q in (3), calculate either

$$\underbrace{\frac{p^n - 1}{p - 1} \times \frac{p^{n+1} - 1}{p^2 - 1} \times \frac{p^{n+2} - 1}{p^3 - 1} \cdots}_{q \text{ times}} \tag{5}$$

or

$$\underbrace{\frac{p^{q+1} - 1}{p - 1} \times \frac{p^{q+2} - 1}{p^2 - 1} \times \frac{p^{q+3} - 1}{p^3 - 1} \cdots}_{(n-1) \text{ times}} \tag{6}$$

and multiply all these numbers.

Remark 2. The numbers (5), (6) are, of course, equal. We may prefer the formula that is 'shorter', that is (5) for $q < n - 1$ and (6) for $n - 1 < q$.

Remark 3. Both formulae have the form of a rational function in p but stand actually for a polynomial. The expressions for its coefficients are, however, complicated.

Example. Determine the number of sublattices of the index 12 of a 5-dimensional lattice. Since $12 = 2^2 \times 3$, the formula (5) is preferred according to Remark 2 for both factors 2^2 and 3. We get

$$\frac{2^5 - 1}{2 - 1} \times \frac{2^6 - 1}{2^2 - 1} \times \frac{3^5 - 1}{3 - 1} = 78\,771.$$

Formula (6) gives the same number as a product of eight fractions where, however, five cancellings occur.

‡ Putting for $n = 1$ the 'empty' product $\prod_{j=1}^{n-1}$ in (4) equal to 1.

For comparison: to use formula (2) would require the construction and summing of 75 products.

3. Proof

It is known (*e.g.* Baake, 1997, Appendix) that

$$F(n, k_1 k_2) = F(n, k_1) F(n, k_2)$$

for positive integers k_1, k_2 without a common factor. Thus, we can confine ourselves in (3) to $m = 1$ and put $p = p_1$, $q = q_1$ so that

$$k = p^q \quad (p > 1 \text{ prime}, q > 0). \tag{7}$$

Further, we keep the prime number p fixed and denote

$$G(n, s) = F(n, p^s) \quad \text{for } n > 0, s \geq 0.$$

Then,

$$G(n, 0) = G(1, s) = 1 \quad \text{for } n > 0, s \geq 0. \tag{8}$$

First we want to prove that $G(n, q)$ $(n > 0, q > 0)$ is equal to the number (5). From (1) and (7), it follows that in any of the matrices of $\mathcal{R}(n, p^q)$ there is

$$r_{11} = p^i,$$

where i is an integer fulfilling $0 \leq i \leq q$. Thus, let us take a certain i with this property and ask how many matrices are there in $\mathcal{R}(n, p^q)$ with $r_{11} = p^i$. Let $\mathbf{M}$ be one of these matrices. If $n = 1$, it is the only one in accordance with (8).

Further, let $n > 1$. Then any of the $n - 1$ elements

$$r_{21}, \ldots, r_{n1}$$

may assume any of the p^i values

$$0, 1, \ldots, p^i - 1$$

which makes

$$p^{i(n-1)} \tag{9}$$

configurations in the first column of $\mathbf{M}$. Any of them is combined with a matrix $\mathbf{M}'$ complementary in $\mathbf{M}$ to the element r_{11}. The product of the elements of $\mathbf{M}'$ in the main diagonal is p^{q-i}. Thus, $\mathbf{M}'$ belongs to the set $\mathcal{R}(n - 1, p^{q-i})$ that contains, according to its definition, $G(n - 1, q - i)$ matrices. Putting this together with (9), we get that the number of matrices in $\mathcal{R}(n, p^q)$ with $r_{11} = p^i$ is equal to

$$p^{i(n-1)} G(n - 1, q - i).$$

Summing over i gives

$$G(n, q) = \sum_{i=0}^{q} p^{i(n-1)} G(n - 1, q - i). \tag{10}$$

Here it is assumed that $n > 1$, $q > 0$ but the formula is valid also for $n > 1$, $q = 0$ according to (8). Writing in (10) $q - 1$ instead of q, we get

$$G(n, q - 1) = \sum_{i=0}^{q-1} p^{i(n-1)} G(n - 1, q - i - 1) \tag{11}$$

for $n > 1$, $q > 0$. From (10) and (11), a recursion formula

$$G(n, q) = G(n - 1, q) + p^{n-1} G(n, q - 1) \tag{12}$$

follows for $n > 1$, $q > 0$. By this formula and the 'boundary' values (8), the function $G(n, s)$ $(n > 0,\ s \geq 0)$ is uniquely determined.

Now we define a function H in this way: $H(n, q)$ is equal to the number (5) for $n > 0$, $q > 0$ and $H(n, 0) = 1$ for $n > 0$. Then, the recursion formula (12) and the relations (8) remain correct if H is written there instead of G. But then G and H must be identical. Thus, $G(n, q)$ is equal to the number (5) *q.e.d.*

Finally, we can directly verify that the numbers (5), (6) are equal and the proof is completed.

4. Concluding remarks

Two alternative formulae for determining the number of sublattices of the index k of an n-dimensional lattice were derived. They enable a comfortable numerical calculation, especially for greater values of n and/or k.

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